L_2-CONVERGENCE OF A NEAREST NEIGHBOR ESTIMATOR OF THE INTENSITY FUNCTION OF A CYCLIC POISSON PROCESS

I W. MANGKU[1]

ABSTRACT. We consider the problem of estimating the intensity function of a cyclic Poisson process. We suppose that only a single realization of the cyclic Poisson process is observed within a bounded 'window', and our aim is to estimate consistently the intensity function at a given point. A nearest neighbor estimator of the intensity function is proposed, and we show that our estimator is L_2-consistent, as the window expands.

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1. INTRODUCTION

Let X be a cyclic Poisson process on the real line $\mathbf{R}$ with (unknown) locally integrable intensity function $\lambda : \mathbf{R} \to \mathbf{R}^+ \cup \{0\}$. In addition, λ is assumed to be cyclic with (unknown) period $\tau \in \mathbf{R}^+$, i.e.

$$\lambda(s + k\tau) = \lambda(s) \tag{1}$$

for all $s \in \mathbf{R}$ and $k \in \mathbf{Z}$, where $\mathbf{Z}$ denotes the set of integers. We do not assume any parametric form of λ.

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the cyclic Poisson process X is observed, though only within a bounded interval $[0, n]$. The aim of this paper is to estimate consistently the intensity function λ at a given point s using an estimator based on nearest neighbor distances, from a single realization $X(\omega)$ of the Poisson process X observed in $[0, n]$.

Let $\hat{\tau}$ be an estimator of the period τ, e.g. the one proposed and studied in [4], or perhaps the estimator of τ investigated by [14]. We assume that the estimator $\hat{\tau} = \hat{\tau}_n$ satisfies the condition $n\,|\hat{\tau}_n - \tau| = o_p\,(k_n/n)$, as $n \to \infty$, with k_n as in (2) and (3).

Let $s_i,\ i = 1, \ldots, X([0, n], \omega)$, denote the locations of the points in the realization $X(\omega)$ of the Poisson process X, observed in window $[0, n]$. Here $X([0, n], \omega)$ is nothing but the cardinality of the data set $\{s_i\}$.

It is well-known (see, e.g. [2]) that, conditionally given $X([0, n]) = m$, $(s_1, \ldots, s_m)$ can be viewed as a random sample of size m from a distribution

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Bogor Agricultural University Jl. Meranti, Kampus IPB Dramaga, Bogor, 16680 Indonesia

with density f, which is given by

$$f(u) = \frac{\lambda(u)}{\int_0^n \lambda(v)dv} \mathbf{I}(u \in [0, n]),$$

while the simultaneous density $f(s_1, \ldots, s_m)$, of $(s_1, \ldots, s_m)$ is given by

$$f(s_1, \ldots, s_m) = \frac{\prod_{i=1}^m \lambda(s_i)}{\left(\int_0^n \lambda(v)dv\right)^m} \mathbf{I}\left((s_1, \ldots, s_m) \in [0, n]^m\right),$$

where $\mathbf{I}$ denotes the indicator function.

Let $\hat{s}_i$, $i = 1, \ldots, m$, denote the location of the point s_i $(i = 1, \ldots, m)$, after translation by a multiple of $\hat{\tau}_n$ such that $\hat{s}_i \in B_{\hat{\tau}_n}(s)$, for all $i = 1, \ldots, m$, where $B_{\hat{\tau}_n}(s) = [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$. The translation can be described more precisely as follows. We cover the window $[0, n]$ by $N_{n,\hat{\tau}_n}$ adjacent disjoint intervals $B_{\hat{\tau}_n}(s + j\hat{\tau}_n)$, for some integer j, and let $N_{n,\hat{\tau}_n}$ denote the number of such intervals, provided $B_{\hat{\tau}_n}(s + j\hat{\tau}_n) \cap [0, n] \neq \emptyset$. Then, for each j, we shift the interval $B_{\hat{\tau}_n}(s + j\hat{\tau}_n)$ (together with the data points of $X(\omega)$ contained in this interval) by the amount $j\hat{\tau}_n$ such that after translation the interval coincide with $B_{\hat{\tau}_n}(s)$.

Let $k = k_n$ be a sequence of positive integers such that

$$k_n \to \infty, \tag{2}$$

and

$$\frac{k_n}{n} \downarrow 0, \tag{3}$$

as $n \to \infty$.

Let now $|\hat{s}_{(k_n)} - s|$ denote the k_n-th order statistics of $|\hat{s}_1 - s|, \ldots, |\hat{s}_m - s|$, given $X([0, n]) = m$. A nearest neighbor estimator for λ at the point s, is given by

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n k_n}{2n|\hat{s}_{(k_n)} - s|} \mathbf{I}(X([0, n]) \geq k_n). \tag{4}$$

2. MAIN RESULTS

Our main results are the following two theorems. The first theorem states that the bias of $\hat{\lambda}_n(s)$ converges to 0 and the second theorem states that the variance of $\hat{\lambda}_n(s)$ converges to 0, $n \to \infty$. These two theorems yield Corollary 2.3, which states that the *MSE* of $\hat{\lambda}_n(s)$ converges to 0, $n \to \infty$.

Theorem 2.1. *Suppose that λ is periodic and locally integrable. If, in addition, (2) and (3) hold true, and*

$$n\,|\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n \frac{k_n}{n}\right) \tag{5}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then

$$\mathbf{E}\hat{\lambda}_n(s) \;\to\; \lambda(s) \tag{6}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Theorem 2.2. *Suppose that λ is periodic and locally integrable. If, in addition, (2), (3) and (5) hold, then*

$$Var\left(\hat{\lambda}_n(s)\right) \;\rightarrow\; 0 \tag{7}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

By Theorems 2.1 and 2.2 we have the following Corollary.

Corollary 2.3. *Suppose that λ is periodic, locally integrable, (2) and (3) hold. If, in addition, (5) holds true, then*

$$MSE\left(\hat{\lambda}_n(s)\right) = Var\left(\hat{\lambda}_n(s)\right) + Bias^2\left(\hat{\lambda}_n(s)\right) \rightarrow 0$$

as $n \to \infty$, for each s at which λ is continuous and positive. In other words, $\hat{\lambda}_n(s)$ is L_2-consistent in estimating $\lambda(s)$.

We remark that nearest neighbor estimators for estimating density functions, was studied by [9], [15], [10], [11], and some others. In the construction of our nearest neighbor estimator (4) we employ the periodicity of λ (cf. (1)) to combine different pieces from our data set, in order to mimic the 'infill asymptotic' framework.

Kernel type estimators for the intensity function λ at a given point s, are proposed and studied by [3], [6] and [7]. In [3] it is proved that their estimator is L_2-consistent, provided λ has a parametric form, while [6] consider a cyclic Poisson process and prove that their estimator is weakly and strongly consistent, provided s is a Lebesgue point of λ. Statistical properties of this estimator are established in [7]. We also refer to [8] and [5] for some related statistical work on Poisson intensity function.

Remark 2.4. Since $\hat{\lambda}_n(s) = 0$ if $X([0,n]) < k_n$, we have that

$$\mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) < k_n) = Var(\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) < k_n)) = 0.$$

This implies $\mathbf{E}\hat{\lambda}_n(s) = \mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) \geq k_n)$, and

$$Var(\hat{\lambda}_n(s)) = Var(\hat{\lambda}_n(s))\mathbf{I}(X([0,n]) \geq k_n).$$

Hence, in all of our proofs in this paper, we only need to consider the case $X([0,n]) \geq k_n$.

3. PROOF OF THEOREM 2.1

By Remark 2.4, the l.h.s. of (6) is equal to

$$\frac{k_n}{2n}\mathbf{E}\frac{\hat{\tau}_n}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n) = \frac{\tau k_n}{2n}\mathbf{E}\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)$$
$$+\frac{k_n}{2n}\mathbf{E}\frac{(\hat{\tau}_n - \tau)}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n). \tag{8}$$

We will prove (6) by showing that the first term on the r.h.s. of (8) is equal to $\lambda(s) + o(1)$ as $n \to \infty$, while its second term is of order $o(1)$ as $n \to \infty$.

First we consider the first term on the r.h.s. of (8). For each n, let A_n denote the set of all integers m_n, where $C_{1,n} \leq m_n \leq C_{2,n}$, with $C_{1,n} = [\theta n - (\theta n)^{1/2} a_n]$, $C_{2,n} = [\theta n - (\theta n)^{1/2} a_n]$ and a_n is an arbitrary sequence such that $a_n \to \infty$ and $a_n = o(n^{1/2})$ as $n \to \infty$. Let $A_n^c = [k_n, \infty) \setminus A_n$. Then, the expectation in the first term on the r.h.s. of (8) can be computed as follows

$$
\begin{aligned}
&= \mathbf{E}\left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)\middle| X([0,n]) = m\right)\right) \\
&= \sum_{m_n \in A_n}\left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X([0,n]) = m_n\right)\right)\mathbf{P}(X([0,n]) = m_n) \\
&+ \sum_{m=k_n}^{C_{1,n}-1}\left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X([0,n]) = m\right)\right)\mathbf{P}(X([0,n]) = m) \\
&+ \sum_{m=C_{2,n}+1}^{\infty}\left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X([0,n]) = m\right)\right)\mathbf{P}(X([0,n]) = m). \quad (9)
\end{aligned}
$$

First we consider the first term on the r.h.s. of (9). To begin with, we first consider this term with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$, where $|\bar{s}_{(k_n)} - s|$ is defined similarly to $|\hat{s}_{(k_n)} - s|$ provided $\hat{\tau}_n$ is replaced by the true period τ. It is well known (see e.g. [12], page 15) that, conditionally given $X([0,n]) = m_n \in A_n$, $|\bar{s}_{(k_n)} - s|$ has the same distribution as $H_n^{-1}(Z_{k_n:m_n})$, where $Z_{k_n:m_n}$ denotes the k_n-th order statistics of a sample $Z_1, \ldots, Z_{m_n}$ of size m_n from the uniform $(0,1)$ distribution. Note that the same device was employed by [11] in his analysis of multivariate nearest neighbor density estimators. First we write the expectation appearing in the first term on the r.h.s. of (9) as

$$
\begin{aligned}
&\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right)\middle| X([0,n]) = m_n\right) \\
&+\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right)\middle| X([0,n]) = m_n\right), \quad (10)
\end{aligned}
$$

for some sequence of positive real numbers $\epsilon_n \downarrow 0$ as $n \to \infty$, and $\tilde{Z}_{k_n:m_n} = Z_{k_n:m_n} - \mathbf{E}Z_{k_n:m_n} = Z_{k_n:m_n} - k_n/(m_n+1)$. Conditionally given $X([0,n]) =$

m_n, we have

$$\begin{aligned}
&|\bar{s}_{(k_n)} - s|\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
&\stackrel{d}{=} \left\{\frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{n}\right) + \left(\frac{\theta\tau}{2\lambda(s)}\right)\tilde{Z}_{k_n:m_n} + o\left(\tilde{Z}_{k_n:m_n}\right)\right\} \\
&\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
&= \left\{\frac{\tau k_n}{2\lambda(s)n} + o\left(\frac{k_n}{n}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right), \qquad (11)
\end{aligned}$$

as $n \to \infty$. Combining (11) and Lemma 4.1 in the Appendix, conditionally given $X([0,n]) = m_n$, we then have

$$\begin{aligned}
&|\hat{s}_{(k_n)} - s|\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
&\stackrel{d}{=} \left\{\frac{\tau k_n}{2\lambda(s)n} + o\left(\frac{k_n}{n}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
&= \left\{\frac{\tau k_n}{2\lambda(s)n}(1+o(1))\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right), \qquad (12)
\end{aligned}$$

as $n \to \infty$. By by Lemma 4.3 in the Appendix, there exists a positive constant C_0 such that

$$\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right) \leq 2\exp\left\{-C_0\epsilon_n^2 k_n\right\} \leq 2\exp\left\{-C_0 k_n^{1/2}\right\}, \qquad (13)$$

as $n \to \infty$, provided $\epsilon_n^{-1} = o(k_n^{1/4})$ as $n \to \infty$. Throughout this proof, we take $\epsilon_n^{-1} = o(k_n^{1/4})$ as $n \to \infty$. From (13), since $k_n \to \infty$ which implies the r.h.s. of (13) is $o(1)$ as $n \to \infty$, we obtain

$$\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) = 1 - o(1), \qquad (14)$$

as $n \to \infty$. By (12) and (14), we can compute the following conditional expectation

$$\begin{aligned}
&\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right)\middle| X([0,n]) = m_n\right) \\
= \;& \mathbf{E}\,\frac{1}{(\tau k_n)(2\lambda(s)n)^{-1}(1+o(1))}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
= \;& \mathbf{E}\,\frac{2\lambda(s)n}{\tau k_n}(1+o(1))\,\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
= \;& \frac{2\lambda(s)n}{\tau k_n} + o\left(\frac{n}{k_n}\right), \qquad (15)
\end{aligned}$$

as $n \to \infty$.

Next we consider the second term of (10). First note that

$$\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right) = \mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right)$$
$$+\mathbf{I}\left(Z_{k_n:m_n} < \frac{k_n}{m_n+1} - \epsilon_n \frac{k_n}{m_n}\right).$$

For the case $Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}$, by Lemma 4.1 in the Appendix, conditionally given $X([0,n]) = m_n$, we have

$$|\hat{s}_{(k_n)} - s| = |\bar{s}_{(k_n)} - s| + o\left(\frac{k_n}{n}\right) = H_n^{-1}\left(Z_{k_n:m_n}\right) + o\left(\frac{k_n}{n}\right)$$
$$\geq H_n^{-1}\left(\frac{k_n}{m_n+1}\right) + o\left(\frac{k_n}{n}\right)$$
$$= H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(n^{-1})\right) + o\left(\frac{k_n}{n}\right)$$
$$= \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{n}\right) \geq \frac{\tau k_n}{4\lambda(s)n},$$

for sufficiently large n. Hence, for sufficiently large n, conditionally given $X([0,n]) = m_n$, we have

$$\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right)$$
$$\leq \frac{4\lambda(s)n}{\tau k_n}\mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right), \tag{16}$$

which in combination with (13), implies

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right)\middle| X([0,n]) = m_n\right)$$
$$= o\left(\frac{n}{k_n}\right) \tag{17}$$

as $n \to \infty$. Next we will show

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(Z_{k_n:m_n} < \frac{k_n}{m_n+1} - \epsilon_n \frac{k_n}{m_n}\right)\middle| X([0,n]) = m_n\right)$$
$$= o\left(\frac{n}{k_n}\right) \tag{18}$$

as $n \to \infty$. By Lemma 4.1 in the Appendix, the fact that $|\bar{s}_{(k_n)} - s| = H_n^{-1}(Z_{k_n:m_n})$, and an application of mean value theorem, together with a little calculation showing that $H_n^{-1'}(\xi_n) = (\theta\tau)(2\lambda(s))^{-1} + o(1)$ as $n \to \infty$, for any (random) point $\xi_n \in (Z_{k_n:m_n}, k_n(m_n+1)^{-1})$, whenever $\mathbf{I}(Z_{k_n:m_n} < k_n(m_n+1)^{-1} - \epsilon_n k_n m_n^{-1}) = 1$, shows that $|\hat{s}_{(k_n)} - s| = ((\theta\tau)(2\lambda(s))^{-1} + o(1))Z_{k_n:m_n} + o(k_n n^{-1})$, as $n \to \infty$. Since $\mathbf{E}Z_{k_n:m_n}^{-2} = \mathcal{O}(m_n^2 k_n^{-2})$ as $n \to \infty$, by an application of Cauchy-Schwarz inequality and (13), we can easily completes the proof of (18). Combining (15), (17) and (18), we have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\middle| X([0,n])=m_n\right)=\frac{2\lambda(s)n}{\tau k_n}+o\left(\frac{n}{k_n}\right) \tag{19}$$

as $n\to\infty$. By an exponential bound for the Poisson probabilities (Lemma 4.2 in the Appendix), we know that

$$\mathbf{P}\left(X([0,n])\in A_n^c\right)\le\mathcal{O}(1)\exp\left(-\frac{a_n^2}{2+o(1)}\right), \tag{20}$$

which is $o(1)$ as $n\to\infty$, since $a_n\to\infty$ as $n\to\infty$. This implies

$$\mathbf{P}\left(X([0,n])\in A_n\right)=(1-o(1)), \tag{21}$$

as $n\to\infty$. By (19) and (21), the first term on the r.h.s. of (9) is equal to

$$\left(\frac{2\lambda(s)n}{\tau k_n}+o\left(\frac{n}{k_n}\right)\right)\mathbf{P}\left(X([0,n])\in A_n\right)=\frac{2\lambda(s)n}{\tau k_n}+o\left(\frac{n}{k_n}\right), \tag{22}$$

as $n\to\infty$.

Next we consider the second and third term on the r.h.s. of (9). First, for any integer $m\in\{[k_n,C_{1,n})\cup(C_{2,n},\infty)\}$, we write the expectation appearing in this term as (10) with m_n replaced by m. For any integer $m\in\{[k_n,C_{1,n})\cup(C_{2,n},\infty)\}$, similar to that in (11) with m_n replaced by m, we have a stochastic expansion for $|\bar{s}_{(k_n)}-s|\mathbf{I}(|\tilde{Z}_{k_n:m}|\le\epsilon_n k_n m^{-1})$, conditionally given $X([0,n])=m$, as follows

$$\begin{aligned}
&|\bar{s}_{(k_n)}-s|\mathbf{I}\left(|\tilde{Z}_{k_n:m}|\le\epsilon_n\frac{k_n}{m}\right)\\
&\stackrel{d}{=}\left\{\frac{\theta\tau k_n}{2\lambda(s)m}+o\left(\frac{k_n}{m}\right)+\mathcal{O}\left(\frac{1}{n}\right)+\mathcal{O}(\tilde{Z}_{k_n:m})\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m}|\le\epsilon_n\frac{k_n}{m}\right)\\
&=\left\{\frac{\theta\tau k_n}{2\lambda(s)m}+o\left(\frac{k_n}{m}\right)+\mathcal{O}\left(\frac{1}{n}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m}|\le\epsilon_n\frac{k_n}{m}\right),
\end{aligned} \tag{23}$$

as $n\to\infty$. Combining (23) and Lemma 4.1 in the Appendix, conditionally given $X([0,n])=m$, we have

$$\begin{aligned}
&|\hat{s}_{(k_n)}-s|\mathbf{I}\left(|\tilde{Z}_{k_n:m}|\le\epsilon_n\frac{k_n}{m}\right)\\
&\stackrel{d}{=}\left\{\frac{\theta\tau k_n}{2\lambda(s)m}+o\left(\frac{k_n}{m}\right)+o\left(\frac{k_n}{n}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m}|\le\epsilon_n\frac{k_n}{m}\right),
\end{aligned} \tag{24}$$

as $n\to\infty$. Note that (13) and (14) remain hold true when $m_n\in A_n$ is now replaced by $m\in\{[k_n,C_{1,n})\cup(C_{2,n},\infty)\}$. By a similar argument as the one used to prove (17) and (18), but with $m_n\in A_n$ replaced by $m\in\{[k_n,C_{1,n})\cup(C_{2,n},\infty)\}$, conditionally given $X([0,n])=m$, we have

$$\mathbf{E}\frac{1}{|\hat{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m}|>\epsilon_n\frac{k_n}{m}\right)=\mathcal{O}\left(\frac{m}{k_n}+\frac{n}{k_n}\right) \tag{25}$$

as $n \to \infty$. Then, by (14) with m_n replaced by $m \in [k_n, C_{1,n})$ and (24), in combination with (25), we have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \middle| X([0,n]) = m \right) = \mathcal{O}\left(\frac{n}{k_n}\right), \tag{26}$$

as $n \to \infty$, uniformly for all $m \in [k_n, C_{1,n})$. By (26), the second term on the r.h.s. of (9) is equal to $\mathcal{O}(nk_n^{-1})\mathbf{P}(X([0,n]) \in [k_n, C_{1,n}))$. Since by (20) we have $\mathbf{P}(X([0,n]) \in [k_n, C_{1,n})) \leq \mathbf{P}(X([0,n]) \in A_n^c) = o(1)$, as $n \to \infty$, this term is of order $o(nk_n^{-1})$ as $n \to \infty$.

For any $m \in (C_{2,n}, \infty)$, since $m > (\theta n) + (\theta n)^{1/2} a_n$ (for some sequence $a_n \to \infty$ and $a_n = o(n^{1/2})$), we may have the absolute value of the third term on the r.h.s. of (24) is bigger than its first term. If the first term on the r.h.s. of (24) is the leading term, a similar argument as the one used to prove (19) shows that

$$\mathbf{E}(|\hat{s}_{(k_n)} - s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})|X([0,n]) = m) = \mathcal{O}(mk_n^{-1}),$$

as $n \to \infty$. If the third term on the r.h.s. of (24) is the leading term, then there exists a sequence $c_n \to 0$ as $n \to \infty$, such that this term can be written as $c_n k_n n^{-1}$ with $|c_n| > (\theta\tau n)/(2\lambda(s)m)$. For this case, a similar argument as the one used to prove (19) shows that $\mathbf{E}(|\hat{s}_{(k_n)} - s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})|X([0,n]) = m) = \mathcal{O}(nk_n^{-1}c_n^{-1})$ as $n \to \infty$. Since $|c_n| > (\theta\tau n)/(2\lambda(s)m)$ which implies $|c_n^{-1}| < (2\lambda(s)m)/(\theta\tau n)$, we also have

$$\mathbf{E}(|\hat{s}_{(k_n)} - s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})|X([0,n]) = m) = \mathcal{O}(mk_n^{-1}),$$

as $n \to \infty$. A similar argument also holds true when the first and third terms on the r.h.s. of (24) are of the same order. Combining this result with (25), uniformly in $m \in (C_{2,n}, \infty)$, we have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \middle| X([0,n]) = m \right) = \mathcal{O}\left(\frac{m}{k_n}\right), \tag{27}$$

as $n \to \infty$. By (27), the third term on the r.h.s. of (9) can be computed as follows

$$\begin{aligned} &\mathcal{O}\left(\frac{1}{k_n}\right) \sum_{m=C_{2,n}+1}^{\infty} m\, \mathbf{P}\left(X([0,n]) = m\right) \\ &= \mathcal{O}\left(\frac{1}{k_n}\right) \mathbf{E}X([0,n])\mathbf{I}(X([0,n]) > C_{2,n}) \\ &\leq \mathcal{O}\left(\frac{1}{k_n}\right) \left(\mathbf{E}X^2([0,n])\right)^{1/2} \mathbf{P}^{1/2}(X([0,n]) > C_{2,n}) = o\left(\frac{n}{k_n}\right), \end{aligned} \tag{28}$$

as $n \to \infty$, because by periodicity of λ we have $(\mathbf{E}X^2([0,n]))^{1/2} = \mathcal{O}(n)$ as $n \to \infty$, and by (20) we have $\mathbf{P}^{1/2}(X([0,n]) > C_{2,n}) \leq \mathbf{P}^{1/2}(X([0,n]) \in A_n^c) = o(1)$, as $n \to \infty$.

Since the first term on the r.h.s. of (9) is equal to the r.h.s. of (22), while the other terms are of order $o(nk_n^{-1})$ as $n \to \infty$, we then have

$$\mathbf{E}\,\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n) = \frac{2\lambda(s)n}{\tau k_n} + o\left(\frac{n}{k_n}\right), \tag{29}$$

as $n \to \infty$, which implies the first term on the r.h.s. of (8) is equal to $\lambda(s) + o(1)$ as $n \to \infty$.

Next we show that the second term on the r.h.s. of (8) is of order $o(1)$ as $n \to \infty$. By (43) and (29), the absolute value of this term does not exceed

$$\begin{aligned}\frac{C\delta_n k_n^2}{2n^3}\mathbf{E}\,\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n) &= \frac{C\delta_n k_n^2}{2n^3}\mathcal{O}\left(\frac{n}{k_n}\right)\\ &= \mathcal{O}\left(\frac{\delta_n k_n}{n^2}\right) = o(1)\end{aligned}$$

as $n \to \infty$. This completes the proof of Theorem 2.1.

4. PROOF OF THEOREM 2.2

By Remark 2.4, we can write

$$\begin{aligned}&Var\left(\hat{\lambda}_n(s)\right) = Var\left(\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) \geq k_n)\right)\\ &= \mathbf{E}\left(\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) \geq k_n)\right)^2 - \left(\mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) \geq k_n)\right)^2. \quad (30)\end{aligned}$$

By Remark 2.4 and Theorem 2.1, we have $\mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X([0,n]) \geq k_n) = \mathbf{E}\hat{\lambda}_n(s) = \lambda(s) + o(1)$ as $n \to \infty$. This implies the second term on the r.h.s. of (30) is equal to $-\lambda^2(s) + o(1)$ as $n \to \infty$. Then, to prove this theorem, it suffices to show that the first term on the r.h.s. of (30) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$. To do this we argue as follows. The first term on the r.h.s. of (30) is equal to

$$\begin{aligned}&\mathbf{E}\left(\frac{\hat{\tau}_n k_n}{2n|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)\right)^2\\ &= \mathbf{E}\left(\frac{\tau k_n}{2n|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)\right)^2\\ &\quad +\mathbf{E}\left(\frac{(\hat{\tau}_n - \tau)k_n}{2n|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)\right)^2\\ &\quad + 2\mathbf{E}\left(\frac{\tau k_n}{2n|\hat{s}_{(k_n)} - s|}\right)\left(\frac{(\hat{\tau}_n - \tau)k_n}{2n|\hat{s}_{(k_n)} - s|}\right)\mathbf{I}(X([0,n]) \geq k_n). \quad (31)\end{aligned}$$

We will show that the first term on the r.h.s. of (31) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$, while its second and third terms are of order $o(1)$ as $n \to \infty$.

First we consider the first term on the r.h.s. of (31). This term can be written as

$$\frac{\tau^2 k_n^2}{4n^2}\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\mathbf{I}(X([0,n])\geq k_n)\right)^2$$
$$= \frac{\tau^2 k_n^2}{4n^2}\mathbf{E}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\mathbf{I}(X([0,n])\geq k_n)\right)^2\middle| X([0,n])=m\right)\right\} \quad (32)$$

Expectation of the quantity within curly brackets on the r.h.s. of (32) can be computed as follows

$$\sum_{m_n\in A_n}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^2\middle| X([0,n])=m_n\right)\right\}\mathbf{P}\left(X([0,n])=m_n\right)$$
$$+\sum_{m=k_n}^{C_{1,n}-1}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^2\middle| X([0,n])=m\right)\right\}\mathbf{P}\left(X([0,n])=m\right)$$
$$+\sum_{C_{2,n}+1}^{\infty}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^2\middle| X([0,n])=m\right)\right\}\mathbf{P}\left(X([0,n])=m\right) \quad (33)$$

First we consider the first term of (33). The expectation appearing in this term can be written as

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^2\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\leq \epsilon_n\frac{k_n}{m_n}\right)\middle| X([0,n])=m_n\right)$$
$$+\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^2\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|> \epsilon_n\frac{k_n}{m_n}\right)\middle| X([0,n])=m_n\right),$$

where ϵ_n a sequence of positive real numbers converging to zero and $\epsilon_n^{-1} = o(k_n^{1/4})$, as $n\to\infty$. By (12) and (14), we can compute the following conditional expectation

$$\begin{aligned}
&\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^2\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\leq \epsilon_n\frac{k_n}{m_n}\right)\middle| X([0,n])=m_n\right)\\
&= \mathbf{E}\,\frac{1}{(\tau^2k_n^2)(4\lambda^2(s)n^2)^{-1}(1+o(1))^2}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\leq \epsilon_n\frac{k_n}{m_n}\right)\\
&= \frac{4\lambda^2(s)n^2}{\tau^2k_n^2}(1+o(1))\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}|\leq \epsilon_n\frac{k_n}{m_n}\right)\\
&= \frac{4\lambda^2(s)n^2}{\tau^2k_n^2}+o\left(\frac{n^2}{k_n^2}\right),
\end{aligned} \quad (34)$$

as $n \to \infty$. By (16) and (13), together with a similar argument as the one used to prove (18), we have

$$\begin{aligned} &\mathbf{E} \quad \left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \right)^2 \mathbf{I} \left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n} \right) \middle| X([0,n]) = m_n \right) \\ &= \quad o\left(\frac{n^2}{k_n^2} \right), \end{aligned} \tag{35}$$

as $n \to \infty$. By (34) and (35), we have

$$\mathbf{E} \left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \right)^2 \middle| X([0,n]) = m_n \right) = \frac{4\lambda^2(s) n^2}{\tau^2 k_n^2} + o\left(\frac{n^2}{k_n^2} \right), \tag{36}$$

as $n \to \infty$. By (36), the first term of (33) is equal to

$$\begin{aligned} &\left(\frac{4\lambda^2(s) n^2}{\tau^2 k_n^2} + o\left(\frac{n^2}{k_n^2} \right) \right) \mathbf{P}(X([0,n]) \in A_n) \\ &= \frac{4\lambda^2(s) n^2}{\tau^2 k_n^2} + o\left(\frac{n^2}{k_n^2} \right), \end{aligned} \tag{37}$$

as $n \to \infty$, because by (21) we have $\mathbf{P}(X([0,n]) \in A_n) = 1 - o(1)$ as $n \to \infty$.

Next we consider the second and third term of (33). By a similar argument as the one used to compute the expectation in (36), but with m_n replaced by m, we have that

$$\mathbf{E} \left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \right)^2 \middle| X([0,n]) = m \right) = \mathcal{O}\left(\frac{n^2}{k_n^2} \right), \tag{38}$$

as $n \to \infty$, uniformly for all $m \in [k_n, C_{1,n})$, and

$$\mathbf{E} \left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \right)^2 \middle| X([0,n]) = m \right) = \mathcal{O}\left(\frac{m^2}{k_n^2} \right), \tag{39}$$

as $n \to \infty$, for each $m \in (C_{2,n}, \infty)$ (cf. also the argument used to prove (27) to handle possibility that the first term on the r.h.s. of (24) is of smaller order than its third term when $m \in (C_{2,n}, \infty)$). By (38), the second term of (33) is equal to $\mathcal{O}(n^2 k_n^{-2}) \mathbf{P}(X([0,n]) \in [k_n, C_{1,n}))$. Since by (20) we have $\mathbf{P}(X([0,n]) \in [k_n, C_{1,n})) \leq \mathbf{P}(X([0,n]) \in A_n^c) = o(1)$, as $n \to \infty$, this term is of order $o(n^2 k_n^{-2})$ as $n \to \infty$. By (39), the third term of (33) can be computed as follows

$$\begin{aligned} &\mathcal{O}\left(\frac{1}{k_n^2} \right) \sum_{m=C_{2,n}+1}^{\infty} m^2 \ \mathbf{P}(X([0,n]) = m) \\ &= \mathcal{O}\left(\frac{1}{k_n^2} \right) \mathbf{E} X^2([0,n]) \mathbf{I}(X([0,n]) > C_{2,n}) \\ &\leq \mathcal{O}\left(\frac{1}{k_n^2} \right) \left(\mathbf{E} X^4([0,n]) \right)^{1/2} \mathbf{P}^{1/2}(X([0,n]) > C_{2,n}) = o\left(n^2 k_n^{-2} \right), \end{aligned}$$

as $n \to \infty$, because by periodicity of λ we have $(\mathbf{E}X^4([0,n]))^{1/2} = \mathcal{O}(n^2)$ as $n \to \infty$, and by (20) we have $\mathbf{P}^{1/2}(X([0,n]) > C_{2,n}) \leq \mathbf{P}^{1/2}(X([0,n]) \in A_n^c) = o(1)$, as $n \to \infty$.

Since the first term of (33) is equal to the r.h.s. of (37), while its second and third terms are of order $o(n^2 k_n^{-2})$ as $n \to \infty$, we then have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)\right)^2 = \frac{4\lambda^2(s)n^2}{\tau^2 k_n^2} + o\left(\frac{n^2}{k_n^2}\right), \tag{40}$$

as $n \to \infty$, which implies the quantity in (32) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$. Hence, the first term on the r.h.s. of (31) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$.

It remains to show that the second and third term on the r.h.s. of (31) are of order $o(1)$ as $n \to \infty$. By (43) and (40), sum of the second term and the absolute value of the third term on the r.h.s. of (31) does not exceed

$$\begin{aligned}&\left(\frac{C^2\delta_n^2 k_n^4}{4n^6} + \frac{C\tau\delta_n k_n^3}{2n^4}\right)\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X([0,n]) \geq k_n)\right)^2 \\ &= \left(\frac{C^2\delta_n^2 k_n^4}{4n^6} + \frac{C\tau\delta_n k_n^3}{2n^4}\right)\mathcal{O}\left(\frac{n^2}{k_n^2}\right) = \mathcal{O}\left(\frac{\delta_n^2 k_n^2}{n^4} + \frac{\delta_n k_n}{n^2}\right) = o(1),\end{aligned}$$

as $n \to \infty$. This completes the proof of Theorem 2.2.

APPENDIX

We begin with a simple lemma, which we will need in our proofs.

Lemma 4.1. *If (3) and (5) hold true, then we have with probability 1 that*

$$|\hat{s}_{(k_n)} - s| = |\bar{s}_{(k_n)} - s| + \mathcal{O}\left(\delta_n \frac{k_n}{n}\right) \tag{41}$$

as $n \to \infty$, provided $X([0,n]) \geq k_n$.

Proof: We can write

$$\begin{aligned}(\hat{s}_{(k_n)} - s) &= (s_{(k_n)} + \hat{\jmath}_{k_n}\hat{\tau}_n - s) = (s_{(k_n)} + \bar{\jmath}_{k_n}\tau - s) + (\hat{\jmath}_{k_n}\hat{\tau}_n - \bar{\jmath}_{k_n}\tau) \\ &= (\bar{s}_{(k_n)} - s) + \hat{\jmath}_{k_n}(\hat{\tau}_n - \tau) + \tau(\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}).\end{aligned} \tag{42}$$

First we will show that the second term on the r.h.s. of (42) is of order $\mathcal{O}(\delta_n k_n n^{-1})$ with probability 1, as $n \to \infty$. To do this, we argue as follows. By (5), there exists a positive constant C such that we have with probability 1

$$|\hat{\tau}_n - \tau| \leq C\delta_n k_n n^{-2}. \tag{43}$$

Since $s \in [0,n]$, by (3) and (43), we have with probability 1 that $|\hat{\jmath}_{k_n}| = \mathcal{O}(n)$ as $n \to \infty$. Combining this order bound and (43), we then have with probability 1 that the second term on the r.h.s. of (42) is of order $\mathcal{O}(\delta_n k_n n^{-1})$ as $n \to \infty$.

Next we will show that the third term on the r.h.s. of (42) is of order $\mathcal{O}(\delta_n k_n n^{-1})$ with probability 1, as $n \to \infty$. Here we only give the proof for the case $\hat{\tau}_n \geq \tau$ and $\hat{\jmath}_{k_n}$, $\bar{\jmath}_{k_n}$ are both positive; because the proofs of the other seven cases are similar. A simple argument shows that

$$\left(\bar{\jmath}_{k_n} - \frac{1}{2}\right) |\hat{\tau}_n - \tau| < (\tau + o_p(1))|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \leq \left(\bar{\jmath}_{k_n} + \frac{1}{2}\right) |\hat{\tau}_n - \tau|. \tag{44}$$

Since $s \in [0, n]$, we have that $\bar{\jmath}_{k_n} = \mathcal{O}([0, n])$ as $n \to \infty$. Then, by (43) and (44), we have with probability 1 that the third term on the r.h.s. of (42) is of order $\mathcal{O}(\delta_n k_n n^{-1})$ as $n \to \infty$. Therefore we have that

$$(\hat{s}_{(k_n)} - s) = (\bar{s}_{(k_n)} - s) + \mathcal{O}\left(\delta_n \frac{k_n}{n}\right)$$

as $n \to \infty$. By the triangle inequality, we have

$$|\bar{s}_{(k_n)} - s| - \left|\mathcal{O}\left(\delta_n k_n n^{-1}\right)\right| \leq |\hat{s}_{(k_n)} - s| \leq |\bar{s}_{(k_n)} - s| + \left|\mathcal{O}\left(\delta_n k_n n^{-1}\right)\right|$$

which implies this lemma. This completes the proof of Lemma 4.1.

Next we present some well-known results which we use in the proofs of our theorems.

Lemma 4.2. *Let X be a Poisson r.v. with $\mathbf{E}X > 0$. Then, for any $\epsilon > 0$, we have*

$$\mathbf{P}\left(\frac{|X - \mathbf{E}X|}{(\mathbf{E}X)^{1/2}} > \epsilon\right) \leq 2\exp\left\{-\frac{\epsilon^2}{2 + \epsilon(\mathbf{E}X)^{-1/2}}\right\}. \tag{45}$$

Proof: We refer to [13].

An exponential bound for 'intermediate' uniform order statistics is given in the following lemma.

Lemma 4.3. *Let k_n and m_n, $n = 1, 2, \ldots$ be sequences of positive integers, and $Z_{k_n:m_n}$ denote the k_n-th order statistic of a random sample of size m_n from the uniform distribution on $(0, 1)$. If $k_n/m_n \downarrow 0$ as $m_n \to \infty$, then for each $\alpha_n > 0$ such that $\alpha_n^{-1} = o(m_n k_n^{-1/2})$ and $\alpha_n = \mathcal{O}(k_n^{1/2})$, there exists a positive absolute constant C_0 and a (large) positive integer n_0 such that*

$$\mathbf{P}\left(\left|Z_{k_n:m_n} - \frac{k_n}{m_n + 1}\right| \left(\frac{m_n}{k_n/(m_n + 1)\,(1 - k_n/(m_n + 1))}\right)^{1/2} \geq \alpha_n\right)$$
$$\leq 2\exp\left\{-C_0\alpha_n^2\right\}, \tag{46}$$

for all $n \geq n_0$.

Proof: A slight modification of the proof of Lemma A2.1. of [1] gives our bound.

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References

[1] Albers, W., Bickel, P. J., and van Zwet, W. R. (1976). Asymptotic expansions for the power of distribution free tests in the one-sample problem. *Ann. Statist.* **4**, 108-156.

[2] Cressie, N. A. C. (1993). *Statistics for Spatial Data.* Revised Edition. Wiley, New York.

[3] Helmers, R., and Zitikis, R. (1999). On estimation of Poisson intensity functions. *Ann. Inst. Stat. Math.* 51, 2, 265-280.

[4] Helmers, R., and Mangku, I W. (2003). On estimating the period of a cyclic Poisson process. *Mathematical Statistics and Applications: Festschrift in honor of Constance van Eeden.* (Editors: Marc Moore, Sorana Froda and Christian Leger), IMS Lecture Notes Series - Monograph Series, Volume **42**, 345-356.

[5] Helmers, R., and Mangku, I W. (2009). Estimating the intensity of a cyclic Poisson process in the presence of linear trend. *Annals Inst. of Statistical Mathematics*, **61** (3), 599-628.

[6] Helmers, R., Mangku, I W., and Zitikis, R. (2003). Consistent estimation of the intensity function of a cyclic Poisson process. *J. Multivariate Anal.*, **84**, 19-39.

[7] Helmers, R., Mangku, I W., and Zitikis, R. (2005). Statistical properties of a kernel-type estimator of the intensity function of a cyclic Poisson process. *J. Multivariate Anal.*, **92**, 1-23.

[8] Helmers, R., Mangku, I W., and Zitikis, R. (2007). A non-parametric estimator of the doubly-periodic Poisson intensity function. *Statistical Methodology*, **4**, 481-492.

[9] Loftsgaarden, D. O., and Quesenberry, C. P. (1965). A nonparametric estimate of a multivariate density function. *Ann. Math. Statist.* **5**, 1049-1051.

[10] Moore, D. S., and Yackel, J. W. (1977). Consistency properties of nearest neighbor density function estimators. *Ann. Statist.* **5**, 143-154.

[11] Ralescu, S. S. (1995). The law of the iterated logarithm for the multivariate nearest neighbor density estimators. *J. Multivariate Anal.* **53**, 159-178.

[12] Reiss, R. D. (1989). *Approximate Distributions of Order Statistics With Applications to Nonparametric Statistics.* Springer-Verlag, New York.

[13] Reiss, R. D. (1993). *A Course on Point Processes.* Springer-Verlag, New York.

[14] Vere-Jones, D. (1982). On the estimation of frequency in point-process data. *J. Appl. Prob.* **19A**, 383-394.

[15] Wagner, T. J. (1973). Strong consistency of a nonparametric estimate of a density function. *IEEE Trans. Systems Man Cybernet.* **3**, 289-290.