Estimation of the Local Intensity of a Cyclic Poisson Process by Means of Nearest Neighbor Distances

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Abstract

We consider the problem of estimating the local intensity of a cyclic Poisson point process, when we know the period. We suppose that only a single realization of the cyclic Poisson point process is observed within a bounded 'window', and our aim is to estimate consistently the local intensity at a given point. A nearest neighbor estimator of the local intensity is proposed, and we show that our estimator is weakly and strongly consistent, as the window expands.

Keywords and Phrases: cyclic Poisson point process, cyclic intensity function, nonparametric estimation, nearest neighbor estimator, period, weak consistency, strong consistency.

1 Introduction

We consider a cyclic Poisson point process X in $\mathbb{R}$ with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν, and with (unknown) locally integrable intensity function $\lambda : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$. In addition, λ is assumed to be cyclic with period $\tau \in \mathbb{R}^+$, i.e.

$$\lambda(s + k\tau) = \lambda(s) \tag{1.1}$$

for all $s \in \mathbb{R}^+$ and $k \in \mathbb{Z}$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let us suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the cyclic Poisson point process X is observed, though only within a bounded interval, called 'window', $W \subset \mathbb{R}$. The aim of this paper is to estimate consistently the intensity function λ at a given point s using an estimator based on nearest neighbor distances, from a single realization $X(\omega)$ of the Poisson process X observed in $W = W_n$, in such a way that

$$|W_n| \to \infty, \tag{1.2}$$

as $n \to \infty$, where $|W_n| = \nu(W_n)$ denotes the size (or the Lebesgue measure) of the window W_n.

Let s_i, $i = 1, \ldots, X(W_n, \omega)$, denote the locations of the points in the realization $X(\omega)$ of the Poisson process X, observed in window W_n. Here $X(W_n, \omega)$ is nothing but the cardinality of the data set $\{s_i\}$.

It is well-known (see, e.g. page 651 of [2]) that, conditionally given $X(W_n) = m$, $(s_1, \ldots, s_m)$ can be viewed as a random sample of size m from a distribution with density f, which is given by

$$f(u) = \frac{\lambda(u)}{\int_{W_n} \lambda(v)dv} I(u \in W_n), \tag{1.3}$$

while the simultaneous density $f(s_1, \ldots, s_m)$, of $(s_1, \ldots, s_m)$ is given by

$$f(s_1, \ldots, s_m) = \frac{\prod_{i=1}^{m} \lambda(s_i)}{\left(\int_{W_n} \lambda(v)dv\right)^m} I((s_1, \ldots, s_m) \in W_n^m). \tag{1.4}$$

Let $\bar{s}_i$, $i = 1, \ldots, m$, denote the location of the point s_i $(i = 1, \ldots, m)$, after translation by a multiple of period τ such that $\bar{s}_i \in B_\tau(s)$, for all $i = 1, \ldots, m$, where $B_\tau(s) = [s - \frac{\tau}{2}, s + \frac{\tau}{2})$. The translation can be described more precisely as follows. We cover the window W_n by $N_{n,\tau}$ adjacent disjoint intervals $B_\tau(s + j\tau)$, for some integer j, and let $N_{n,\tau}$ denote the number of such intervals, provided $B_\tau(s + j\tau) \cap W_n \neq \emptyset$. Then, for each j, we shift the interval $B_\tau(s + j\tau)$ (together with the data points of $X(\omega)$ contained in this interval) by the amount $j\tau$ such that after translation the interval coincide with $B_\tau(s)$.

By periodicity of λ, we have that $\lambda(\bar{s}_i) = \lambda(s_i)$, for each $i = 1, \ldots, X(W_n, \omega)$. For any $A \subset B_\tau(s)$, let $\bar{X}_n(A)$ denotes the number of points $\bar{s}_i$ in A. Then, of course, $\bar{X}_n(B_\tau(s)) = X(W_n)$, where $\bar{X}_n$ is a Poisson process with intensity function

$$\lambda_n(u) = \lambda(u) \sum_{j=-\infty}^{\infty} I(u + j\tau \in W_n)$$

(cf. [5], Superposition Theorem and Restriction Theorem, page 16-17). As a result, (cf. (1.3) and (1.4)), conditionally given $\bar{X}_n(B_\tau(s)) = m$, $(\bar{s}_1, \ldots, \bar{s}_m)$ can be viewed as a random sample of size m from a distribution with density $\bar{f}$, which is given by

$$\bar{f}(u) = \frac{\lambda_n(u)}{\int_{W_n} \lambda(v)dv} I(u \in B_\tau(s)) = \frac{\lambda_n(u)}{\int_{B_\tau(s)} \lambda_n(v)dv} I(u \in B_\tau(s)), \tag{1.5}$$

while the simultaneous density $\bar{f}(\bar{s}_1, \ldots, \bar{s}_m)$, of $(\bar{s}_1, \ldots, \bar{s}_m)$ is given by

$$\bar{f}(\bar{s}_1, \ldots, \bar{s}_m) = \frac{\prod_{i=1}^{m} \lambda_n(\bar{s}_i)}{\left(\int_{W_n} \lambda(v)dv\right)^m} I((\bar{s}_1, \ldots, \bar{s}_m) \in B_\tau(s)^m). \tag{1.6}$$

For any real number $x \geq 0$, define

$$\begin{aligned} H_n(x) &= \mathbf{P}(|\bar{s}_i - s| \leq x \ |X(W_n) = m) \\ &= \mathbf{P}(s - x \leq \bar{s}_i \leq s + x \ |X(W_n) = m) \\ &= \int_{s-x}^{s+x} \frac{\lambda_n(u)}{\int_{W_n} \lambda(v)dv} \mathbf{I}(u \in B_\tau(s))du. \end{aligned} \tag{1.7}$$

2 Main Results

Let $k = k_n$ be a sequence of positive integers such that

$$k_n \to \infty, \tag{2.1}$$

and

$$\frac{k_n}{|W_n|} \downarrow 0, \tag{2.2}$$

as $n \to \infty$.

Now we consider the order statistics of the random sample $|\bar{s}_1 - s|, \ldots, |\bar{s}_m - s|$ of size m from H_n. Let $|\bar{s}_{(k)} - s|$ denote the k-th order statistics of the sample $|\bar{s}_1 - s|, \ldots, |\bar{s}_m - s|$, given $X(W_n) = m$. A nearest neighbor estimator for λ at the point s, is given by

$$\hat{\lambda}_n(s) = \frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|}. \tag{2.3}$$

Remark: Note that $\hat{\lambda}_n(s)$ is well-defined provided $k_n \leq X(W_n)$. Since

$$\mathbf{P}(k_n \leq X(W_n)) = \mathbf{P}(k_n/|W_n| \leq X(W_n)/|W_n|) \to 1,$$

as $|W_n| \to \infty$, (because of (2.2) and the fact that $X(W_n)/|W_n| \xrightarrow{p} \theta$, with $\theta > 0$, where $\theta = \tau^{-1}\int_0^\tau \lambda(s)ds$, the 'global intensity' of X), we can conclude no matter how we define $\hat{\lambda}_n(s)$ in case $k_n > X(W_n)$, Theorem 1 remains valid. To check that the above conclusion also holds for Theorem 2, we need to show that

$$\sum_{n=1}^{\infty} \mathbf{P}(k_n > X(W_n)) < \infty$$

But, by (2.2), the exponential bound for Poisson probabilities (see Lemma 1 in section 5), and (2.5), it is easy to show that $\mathbf{P}(k_n > X(W_n))$ is summable.

Theorem 1 *Suppose that λ is periodic with period τ and locally integrable. If, in addition (2.1) and (2.2) hold, then*

$$\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s), \tag{2.4}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Throughout the paper, for any random variables Y_n and Y, we write $Y_n \xrightarrow{c} Y$ to denote that Y_n *converges completely* to Y, as $n \to \infty$.

Theorem 2 *Suppose that λ is periodic with period τ and locally integrable. If, in addition*

$$\sum_{n=1}^{\infty} \exp(-\epsilon k_n) < \infty, \tag{2.5}$$

for each $\epsilon > 0$ and (2.2) holds, then

$$\hat{\lambda}_n(s) \xrightarrow{c} \lambda(s), \tag{2.6}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

We remark that nearest neighbor estimators for estimating density functions, was studied by [6], [12], [7], [8], and some others. The condition (2.5) also appears in [12]. In the construction of our nearest neighbor estimator (2.3) we employ the periodicity of λ (cf. (1.1)) to combine different pieces from our data set, in order to mimic the 'infill asymptotic' framework.

Kernel type estimators for the intensity function λ at a given point s, are proposed and studied by [3] and [4]. [3] show that their estimator is L_2-consistent, provided λ has a parametric form, while [4] consider a cyclic Poisson process and prove that their estimator is weakly and strongly consistent, provided s is a Lebesgue point of λ.

3 Proof of Theorem 1

In view of Remark following 2.3, we may assume, without loss of generality, that $k_n \leq X(W_n)$. To prove (2.4), we must show that,

$$\mathbf{P}\left(\left|\frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|} - \lambda(s)\right| \geq \epsilon\right) \to 0 \tag{3.7}$$

as $n \to \infty$, for each sufficiently small $\epsilon > 0$. Choose $\epsilon < \lambda(s)$. Then, a simple calculation shows that, the probability on the l.h.s. of (3.7) is equal to

$$\begin{aligned} &\mathbf{P}\left(\frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} \leq |\bar{s}_{(k_n)} - s| \text{ or } \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)} \geq |\bar{s}_{(k_n)} - s|\right) \\ \leq\ &\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) \\ &+\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right). \end{aligned} \tag{3.8}$$

Then, to prove (3.7), it suffices to check that

$$\mathrm{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) \to 0 \tag{3.9}$$

and

$$\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \le \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right) \to 0 \tag{3.10}$$

as $n \to \infty$, for each $\epsilon > 0$. Here we only give proof of (3.9), because the proof of (3.10) is similar.

Recall $X(W_n)$ is a Poisson with

$$\mathbf{E}X(W_n) = Var(X(W_n)) = \int_{W_n} \lambda(s)ds.$$

Since λ is cyclic (with period τ); we have that

$$\int_{W_n} \lambda(s)ds = \theta|W_n| + \mathcal{O}(1),$$

as $n \to \infty$. Let

$$C_{1,n} = [\theta|W_n| - (\theta|W_n|)^{1/2}a_n] \tag{3.11}$$
$$C_{2,n} = [\theta|W_n| + (\theta|W_n|)^{1/2}a_n], \tag{3.12}$$

where a_n is an arbitrary sequence such that $a_n \to \infty$ and $a_n = o(|W_n|^{1/2})$, as $n \to \infty$. Then, we can write the probability on the l.h.s. of (3.9) as

$$\begin{aligned}
&\sum_{m=k_n}^{\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \ge \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\, |X(W_n) = m\right) \mathbf{P}\left(X(W_n) = m\right) \\
\le\ & \sum_{m=k_n}^{C_{1,n}-1} \mathbf{P}\left(X(W_n) = m\right) + \sum_{m=C_{2,n}+1}^{\infty} \mathbf{P}\left(X(W_n) = m\right) \\
+\ & \max_{C_{1,n} \le m \le C_{2,n}} \mathbf{P}\left(X(W_n) = m\right) \cdot \\
& \cdot \sum_{m=C_{1,n}}^{C_{2,n}} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \ge \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} |X(W_n) = m\right).
\end{aligned} \tag{3.13}$$

It suffices now to show that each term on the r.h.s. of (3.13) converges to zero, as $n \to \infty$.

First we show that the first term on the r.h.s. of (3.13) is $o(1)$, as $n \to \infty$. Since $|\mathbf{E}X(W_n) - \theta|W_n|| = O(1)$, as $n \to \infty$, this quantity is equal to

$$\begin{aligned}
&\mathbf{P}\left(X(W_n) \le C_{1,n} - 1\right) \le \mathbf{P}\left(X(W_n) \le \theta|W_n| - (\theta|W_n|)^{1/2}a_n\right) \\
\le\ & \mathbf{P}\left(|X(W_n) - \mathbf{E}X(W_n)| \ge (\theta|W_n|)^{1/2}a_n - |\mathbf{E}X(W_n) - \theta|W_n||\right) \\
=\ & \mathbf{P}\left((\mathbf{E}X(W_n))^{-1/2}|X(W_n) - \mathbf{E}X(W_n)| \ge \mathcal{O}(1)a_n\right) \\
\le\ & \mathcal{O}(1)\exp\left(-\frac{a_n^2}{2+o(1)}\right),
\end{aligned} \tag{3.14}$$

which is $o(1)$, since $a_n \to \infty$, as $n \to \infty$. Here we have used Lemma 1 of the Section 5. A similar argument also shows that the second term on the r.h.s. of (3.13) is $o(1)$, as $n \to \infty$.

Next we prove that the third term on the r.h.s. of (3.13) is $o(1)$, as $n \to \infty$. Let $m = m_n$ be a positive integer, such that $C_{1,n} \leq m_n \leq C_{2,n}$. Then $m_n \sim \theta|W_n|$, which implies that $k_n/m_n = o(1)$, as $n \to \infty$ (by (2.2)). Recall that $X(W_n)$ has a Poisson distribution with parameter $\mu(W_n) = \int_{W_n} \lambda(s)ds$. A simple calculation, using Stirling's formula, shows that

$$\max_{m_n, C_{1,n} \leq m_n \leq C_{2,n}} P(X(W_n) = m_n = \mathcal{O}(|W_n|^{-1/2}),$$

as $n \to \infty$. It is well-known (see, e.g. [9], p. 15) that, conditionally given $\bar{X}_n(\bar{B}_\tau(s)) = X(W_n) = m_n$, $|\bar{s}_{(k_n)} - s|$ has exactly the same distribution as

$H_n^{-1}(Z_{k_n:m_n})$, where $Z_{k_n:m_n}$ is the k_n-th order statistics of a sample $Z_1, \ldots, Z_{m_n}$ of size m_n from the uniform $(0,1)$ distribution. (We remark in passing that $k_n \leq m_n$ for all n sufficiently large). Note that a similar device was employed by [8] in his analysis of multivariate nearest neighbor density estimators. As a result, the third term on the r.h.s. of (3.13) is equal to

$$\mathcal{O}(|W_n|^{-1/2}) \sum_{m_n = C_{1,n}}^{C_{2,n}} P\left(H_n^{-1}(Z_{k_n:m_n}) \geq \frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)}\right). \tag{3.15}$$

First note that, by choosing $\epsilon < \lambda(s)$, we have

$$\frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)} = \frac{\tau k_n}{2\lambda(s)|W_n|\left(1 - \frac{\epsilon}{\lambda(s)}\right)} \geq \frac{\tau k_n}{2\lambda(s)|W_n|}\left(1 + \frac{\epsilon}{\lambda(s)}\right)$$

$$= \frac{\tau k_n}{2\lambda(s)|W_n|} + \frac{\tau\epsilon k_n}{2\lambda^2(s)|W_n|}. \tag{3.16}$$

We know that, for each m_n,

$$\mathbf{E}Z_{k_n:m_n} = k_n/(m_n + 1)$$

and

$$Var(Z_{k_n:m_n}) = \mathcal{O}(k_n/(m_n^2)).$$

We now need a stochastic expansion for $H_n^{-1}(Z_{k_n:m_n})$. First we simplify the r.h.s. of (1.7) to get for any $x \geq 0$

$$\begin{aligned} H_n(x) &= \frac{(|W_n|/\tau + \mathcal{O}(1))}{(\theta|W_n| + \mathcal{O}(1))} \int_{s-x}^{s+x} \lambda(u) I(u \in \bar{B}_\tau(s)) du \\ &= \left(\frac{1}{\theta\tau} + \mathcal{O}(|W_n|^{-1})\right) \int_{s-x}^{s+x} \lambda(u) I(u \in \bar{B}_\tau(s)) du \\ &= \frac{1}{\theta\tau} \int_{s-x}^{s+x} \lambda(u) I(u \in \bar{B}_\tau(s)) du + \mathcal{O}(|W_n|^{-1}), \end{aligned} \tag{3.17}$$

as $n \to \infty$, uniformly in x. This because $\int_{s-x}^{s+x} \lambda(u) I(u \in \bar{B}_\tau(s)) du \le \theta\tau$. Define function $H(x)$, which is equal to the first term on the r.h.s. of (3.17) for $x \ge 0$, and zero otherwise. The density h of H is given by

$$\begin{aligned} h(x) &= \frac{\lambda(s+x)\mathbf{I}(s+x \in B_\tau(s))}{\theta\tau} \\ &\quad + \frac{\lambda(s-x)\mathbf{I}(s-x \in B_\tau(s))}{\theta\tau}, \end{aligned} \tag{3.18}$$

for any $x > 0$, while $h(0)$ denote the right hand derivative of H at zero. Next note that

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n}) &= \inf\{x : H_n(x) > Z_{k_n:m_n}\} \\ &= \inf\{x : H(x) > Z_{k_n:m_n} + \mathcal{O}(|W_n|^{-1})\} \\ &= H^{-1}\left(Z_{k_n:m_n} + \mathcal{O}(|W_n|^{-1})\right), \end{aligned} \tag{3.19}$$

as $n \to \infty$. Here and elsewhere in this paper we define $H^{-1}(t) = \inf\{x : H(x) > t\}$, $0 \le t < 1$. Now we compute $H^{-1}(0)$. Since $\lambda(s) > 0$ and λ is continuous at s, we see from the first term on the r.h.s. of (3.17) that $H(x) > 0$, while $x > 0$. In other words, the first term on the r.h.s. of (3.17) is equal to zero, if and only if, $x = 0$. Hence $H^{-1}(0) = 0$. Since h is right continuous at 0, the first (right hand) derivative of H^{-1} at 0 can be computed as

$$H^{-1'}(0) = \frac{1}{h(H^{-1}(0))} = \frac{1}{h(0)} = \frac{\theta\tau}{2\lambda(s)}. \tag{3.20}$$

Since $H^{-1'}(0)$ is finite, by Young's form for Taylor's theorem (see [11], p. 45), we can write

$$\begin{aligned} & H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) \\ = \; & H^{-1}(0) + \left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) H^{-1'}(0)(1+o(1)) \\ = \; & \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right), \end{aligned} \tag{3.21}$$

as $n \to \infty$. Because λ is continuous at s, we have

$$\begin{aligned} & H^{-1'}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) = \frac{1}{h\left(H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right)\right)} \\ = \; & \frac{1}{h(|o(1)|)} = \frac{\theta\tau}{2\lambda(s+|o(1)|)} = \frac{\theta\tau}{2\lambda(s)} + o(1), \end{aligned} \tag{3.22}$$

as $n \to \infty$.

Let $\tilde{Z}_{k_n:m_n} = Z_{k_n:m_n} - EZ_{k_n:m_n} = Z_{k_n:m_n} - k_n/(m_n+1)$. Let us write

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n}) &= H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n) \\ &+ H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n), \end{aligned}$$

where ϵ_n is a sequence of positive real numbers such that $\epsilon_n \downarrow 0$ as $n \to \infty$.. Because

$$H^{-1'}(k_n/(m_n+1) + \mathcal{O}(|W_n|^{-1})) = \mathcal{O}(1),$$

as $n \to \infty$, by Young's form for Taylor's theorem, we can write $H_n^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n)$ as (cf. (3.19))

$$\begin{aligned}
& H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n) \\
= \; & H^{-1}\left(Z_{k_n:m_n} + \mathcal{O}(|W_n|^{-1})\right) \mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n) \\
= \; & \left\{ H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) \right. \\
& + \left(Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right) H^{-1'}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) \\
& \left. + o\left(Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right)\right\} \mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n),
\end{aligned} \tag{3.23}$$

as $n \to \infty$. Substituting (3.21) and (3.22) into the r.h.s. of (3.23), we then have

$$\begin{aligned}
H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n) = & \left\{ \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right) \right. \\
+ \left(\frac{\theta\tau}{2\lambda(s)}\right) \tilde{Z}_{k_n:m_n} + & \left. o\left(\tilde{Z}_{k_n:m_n}\right)\right\} \mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n),
\end{aligned} \tag{3.24}$$

as $n \to \infty$. Since $m_n \ge C_{1,n}$, the first term on the r.h.s. of (3.24) does not exceed

$$\begin{aligned}
& \frac{\theta\tau k_n}{2\lambda(s)\left([\theta|W_n| - (\theta|W_n|)^{1/2}a_n] + 1\right)} \le \frac{\theta\tau k_n}{2\lambda(s)\left(\theta|W_n| - (\theta|W_n|)^{1/2}a_n\right)} \\
= \; & \frac{\theta\tau k_n}{2\theta\lambda(s)|W_n|\left(1 - (\theta|W_n|)^{-1/2}a_n\right)} = \frac{\tau k_n}{2\lambda(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right),
\end{aligned} \tag{3.25}$$

as $n \to \infty$. Combining (3.24), (3.25), and (3.16), and by noting also that the first term on the r.h.s. of (3.25) cancels with the first term on the r.h.s. of (3.16), we find that, for sufficiently large n, the probability appearing in (3.15) does not exceed

$$\begin{aligned}
& \mathbf{P}\left(\frac{\theta\tau}{\lambda(s)}|\tilde{Z}_{k_n:m_n}|\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n) + \left|o\left(\tilde{Z}_{k_n:m_n}\right)\right| \mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n) \right. \\
& \left. + H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n) > \frac{\tau\epsilon k_n}{4\lambda^2(s)|W_n|}\right) \\
\le \; & \mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right) + \mathbf{P}\left(|o(\tilde{Z}_{k_n:m_n})| > \frac{\tau\epsilon k_n}{12\lambda^2(s)|W_n|}\right) \\
& + \mathbf{P}\left(H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n) > \frac{\tau\epsilon k_n}{12\lambda^2(s)|W_n|}\right).
\end{aligned} \tag{3.26}$$

First note that, for sufficiently large n, the second term on the r.h.s. of (3.26) does not exceed its first term. Now we notice that $H_n^{-1}(Z_{k_n:m_n}) \le H_n^{-1}(1) = \frac{\tau}{2}$.

Then we find that the third probability on the r.h.s. of (3.26) does not exceed $P(|\tilde{Z}_{k_n:m_n}| > \epsilon_n)$. For convenience we take $\epsilon_n = (\epsilon k_n)/(12\theta\lambda(s)|W_n|)$. Then, the r.h.s. of (3.26) does not exceed

$$3P\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right).$$

Therefore, for sufficiently large n, the quantity in (3.15) does not exceed

$$\begin{aligned} &\mathcal{O}(|W_n|^{-1/2})(C_{2,n} - C_{1,n} + 1)\,P\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right) \\ \leq\ &\mathcal{O}(1)a_n P\left(\left|Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right| \geq \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right), \end{aligned} \tag{3.27}$$

as $n \to \infty$. By Chebyshev's inequality, we find that the probability on the r.h.s. of (3.27) is of order $O(k_n^{-1})$, as $n \to \infty$. By (2.1) and choosing now $a_n = o(k_n)$, as $n \to \infty$, we have that the r.h.s. of (3.27) is $o(1)$ as $n \to \infty$. Hence (3.9) is proved. This completes the proof of Theorem 1.

4 Proof of Theorem 2

To establish (2.6), we must show that

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|} - \lambda(s)\right| \geq \epsilon\right) < \infty, \tag{4.1}$$

for each $\epsilon > 0$. By (3.8), to prove (4.1) it suffices to show, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)}\right) < \infty, \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} P\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s) + \epsilon)}\right) < \infty. \tag{4.3}$$

Here we only give the proof of (4.2), because the proof of (4.3) is similar. To prove (4.2), it suffices to show that, each of the terms on the r.h.s. of (3.13) converges completely to zero, as $n \to \infty$.

Let $C_{1,n}$ and $C_{2,n}$ be as given in (3.11) and (3.12). In order to deal with the first and second term of (3.13), the sequence a_n will now have to satisfy, in addition to the assumption $a_n = o(|W_n|^{1/2})$ which was already needed in the proof of Theorem 1, the additional requirement

$$\sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty.$$

The argument given in (3.14) will then imply that these terms converge completely to zero, as $n \to \infty$.

It remains to show that the third term on the r.h.s. of (3.13) also converges completely to zero, as $n \to \infty$. To do this, it is clear from the proof of Theorem 1, that it suffices now to check that the r.h.s. of (3.27) is summable, for each $\epsilon > 0$.

Let us now consider the probability appearing on the r.h.s. of (3.27). For sufficiently large n, by Lemma 2 (see Section 5), there exists a positive constant C_0 such that the probability on the r.h.s. of (3.27) does not exceed

$$2\exp\left\{-C_0 t_n^2\right\},$$

where

$$t_n = \left(\frac{m_n}{k_n/(m_n+1)\,(1-k_n/(m_n+1))}\right)^{1/2} \frac{k_n\,\epsilon}{12\theta\lambda(s)|W_n|}$$

which (for sufficiently large n) can be replaced with impunity by $\epsilon k_n^{\frac{1}{2}}/(24\lambda(s))$. Hence, for sufficiently large n, the r.h.s. of (3.27) does not exceed

$$\begin{aligned}
& \mathcal{O}(1) a_n \exp\left\{-\frac{C_0\epsilon^2}{576(\lambda(s))^2}k_n\right\} \\
= \;& \mathcal{O}(1)\exp\left\{\log a_n - \frac{C_0\epsilon^2}{1152(\lambda(s))^2}k_n\right\}\exp\left\{-\frac{C_0\epsilon^2}{1152(\lambda(s))^2}k_n\right\} \\
= \;& \mathcal{O}(1)\exp\left\{-\frac{C_0\epsilon^2}{1152(\lambda(s))^2}k_n\right\}, \qquad (4.4)
\end{aligned}$$

provided we require a_n to satisfy $\log a_n = o(k_n)$, as $n \to \infty$. Note that, e.g. the choice $a_n = (k_n)^{1/2}$ satisfies each of the three conditions imposed on a_n, namely $a_n = o(|W_n|^{1/2})$, $\sum_{n=1}^{\infty}\exp(-a_n^2/3) < \infty$, and $\log a_n = o(k_n)$, provided (2.2) and (2.5). By assumption (2.5), we have that the r.h.s. of (4.4) is summable. Hence (4.2) is proved. This completes the proof of Theorem 2.

5 Technical Lemmas

In this section we presents two well-known results which we used in the proof of Theorem 1 dan Theorem 2.

Lemma 1 *Let* X *be a Poisson random variable with* $\mathbf{E}X > 0$. *Then for any* $\epsilon > 0$ *we have*

$$\mathbf{P}\left(\frac{|X-\mathbf{E}X|}{(\mathbf{E}X)^{1/2}} > \epsilon\right) \le 2\exp\left\{-\frac{\epsilon^2}{2+\epsilon(\mathbf{E}X)^{-1/2}}\right\}. \qquad (5.1)$$

Proof: We refer to Reiss ([10], p. 222).

An exponential bound for 'intermediate' uniform order statistics is given in the following lemma.

Lemma 2 *Let k_n and m_n, $n = 1, 2, \ldots$ be sequences of positive integers, and $Z_{k_n:m_n}$ denote the k-th order statistic of a random sample of size m_n from the uniform distribution on (0,1). If $k_n/m_n \downarrow 0$ as $m_n \to \infty$, then for each $\alpha_n > 0$ such that $\alpha_n^{-1} = o(m_n k_n^{-1/2})$ and $\alpha_n = O(k_n^{1/2})$, there exists a positive absolute constant C_0 and a (large) positive integer n_0 such that*

$$\mathbf{P}\left(\left|Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right|\left(\frac{m_n}{k_n/(m_n+1)(1-k_n/(m_n+1))}\right)^{1/2} \geq \alpha_n\right)$$
$$\leq \ 2\exp\left\{-C_0\alpha_n^2\right\}. \tag{5.2}$$

for all $n \geq n_0$.

Proof. A slight modification of the proof of Lemma A2.1. of [1] gives our bound.

■

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